Supplemental Material: Efficient Derivative Computation for Cumulative B-Splines on Lie Groups

8. Appendix

8.1. Used symbols

In general, we use lowercase bold symbols for vectors in $\mathbb{R}^d$, uppercase regular symbols for elements in the Lie group $\mathcal{L}$ and the Lie algebra $\mathcal{A}$, and lowercase regular symbols for scalars, subscripts and superscripts.

8.1.1 Symbols with specific meaning

Some symbols appear repeatedly throughout the paper and have a dedicated meaning, most important:

$\mathbf{p}(u), \mathbf{p}_i$	spline value and control points in $\mathbb{R}^d$
$X(u), X_i$	spline value and control points in $\mathcal{L}$
$R(u), R_i$	spline value and control points in $SO(3)$
$\boldsymbol{\omega}$	velocity
$\boldsymbol{\delta}$	small increment

8.1.2 Indices

While some of the used subscripts and superscripts are just dummy indices, others have a specific meaning that does not change throughout the paper:

k	order of the spline
i	index of control points/time intervals
j	ranges from 0 or 1 to $k-1$, recursion index
l, m, n, s	dummy indices without particular meaning, used for definitions and proofs

8.2. Right Jacobian for $SO(3)$

If $\mathcal{L} = SO(3)$, the right Jacobian and its inverse can be found in [3, p. 40]:

$$J_{\mathrm{r}}(\mathbf{x}) = \mathbb{1} - \frac{1-\cos(\|\mathbf{x}\|)}{\|\mathbf{x}\|^2}\mathbf{x}_\wedge + \frac{\|\mathbf{x}\| - \sin(\|\mathbf{x}\|)}{\|\mathbf{x}\|^3}\mathbf{x}_\wedge^2 , \tag{72}$$

$$J_{\mathrm{r}}(\mathbf{x})^{-1} = \mathbb{1} + \frac{1}{2}\mathbf{x}_\wedge + \left(\frac{1}{\|\mathbf{x}\|^2} - \frac{1+\cos(\|\mathbf{x}\|)}{2\|\mathbf{x}\|\sin(\|\mathbf{x}\|)}\right)\mathbf{x}_\wedge^2 . \tag{73}$$

8.3. Third order time derivatives

For completeness, we state the third order time derivatives for a general Lie group $\mathcal{L}$ here. The proofs are in analogy to those of the first and second order time derivatives, thus we do not repeat them here.

$$\dddot{X} = X\left((\boldsymbol{\omega}_\wedge^{(k)})^3 + 2\boldsymbol{\omega}_\wedge^{(k)}\dot{\boldsymbol{\omega}}_\wedge^{(k)} + \dot{\boldsymbol{\omega}}_\wedge^{(k)}\boldsymbol{\omega}_\wedge^{(k)} + \ddot{\boldsymbol{\omega}}_\wedge^{(k)}\right) , \tag{74}$$

$$\begin{aligned}\ddot{\boldsymbol{\omega}}^{(j)} &= \mathrm{Adj}_{A_{j-1}^{-1}}\, \ddot{\boldsymbol{\omega}}^{(j-1)} + \dddot{\lambda}_{j-1}\mathbf{d}_{j-1} \\ &+ \left[\ddot{\lambda}_{j-1}\boldsymbol{\omega}_\wedge^{(j)} + 2\dot{\lambda}_{j-1}\dot{\boldsymbol{\omega}}_\wedge^{(j)} - \dot{\lambda}_{j-1}^2[\boldsymbol{\omega}_\wedge^{(j)}, D_{j-1}], D_{j-1}\right]_\vee ,\end{aligned} \tag{75}$$

$$\ddot{\boldsymbol{\omega}}^{(1)} = \mathbf{0} \in \mathbb{R}^d . \tag{76}$$

$\ddot{\boldsymbol{\omega}}$ is called *jerk*. For $\mathcal{L} = SO(3)$, the expression (75) becomes slightly simpler:

$$\begin{aligned}\ddot{\boldsymbol{\omega}}^{(j)} &= A_{j-1}^\top \ddot{\boldsymbol{\omega}}^{(j-1)} + \dddot{\lambda}_{j-1}\mathbf{d}_{j-1} \\ &+ \left(\ddot{\lambda}_{j-1}\boldsymbol{\omega}^{(j)} + 2\dot{\lambda}_{j-1}\dot{\boldsymbol{\omega}}^{(j)} - \dot{\lambda}_{j-1}^2\boldsymbol{\omega}^{(j)} \times \mathbf{d}_{j-1}\right) \times \mathbf{d}_{j-1} .\end{aligned} \tag{77}$$

8.4. Complexity analysis

While we are not the first to write down temporal derivatives of Lie group splines, we actually are the first to compute these in only $\mathcal{O}(k)$ matrix operations (multiplications and additions). Additionally, to the best of our knowledge, we are the first to explicitly propose a scheme for Jacobian computation in $SO(3)$, which also does not need more than $\mathcal{O}(k)$ matrix operations. In Table 4, we provide an overview of the needed number of multiplications and additions for the temporal derivatives (both in related work and according to the proposed formulation).

8.5. Proofs

8.5.1 Proof of (20) (cumulative blending matrix)

After substituting summation indices $s \leftarrow k-1-s$ and $l \leftarrow l-s$ we get

$$\widetilde{m}_{0,n}^{(k)} = \frac{C_{k-1}^n}{(k-1)!}\sum_{s=0}^{k-1}\sum_{l=0}^{s}(-1)^l C_k^l (s-l)^{k-1-n} . \tag{78}$$

We now show by induction over k that $\widetilde{m}_{0,n}^{(k)} = \delta_{n,0}$ for all $n = 0, ..., k-1$: for $k=1$, $\widetilde{m}_{0,n}^{(k)} = 1 = \delta_{0,0}$ is trivial. Now, assume $\widetilde{m}_{0,n}^{(k)} = \delta_{n,0}$ for some k.

Starting from the induction assumption

$$\sum_{s=0}^{k-1}\sum_{l=0}^{s}(-1)^l C_k^l (s-l)^{k-1-n} = (k-1)!\,\delta_{n,0} \tag{79}$$

	$\dot{X}$ Baseline		$\dot{X}$ Ours		$\ddot{X}$ Baseline		$\ddot{X}$ Ours, any $\mathcal{L}$		$\ddot{X}$ Ours, $SO(3)$	
m-m mult.	$(k-1)^2+1$	10	1	1	$\frac{1}{2}k^2(k-1)$	24	$2k$	8	2	2
m-v mult.	0	0	$k-1$	3	0	0	$k-1$	3	$2(k-1)$	6
add.	$k-2$	2	$k-1$	3	$\frac{1}{2}k^2(k-1)$	24	$3k-2$	10	$2k-1$	7

Table 4. Number of matrix operations needed to compute temporal derivatives of the $\mathcal{L}$-valued splines: *m-m/m-v mult.* denote matrix-matrix and matrix-vector multiplications, respectively. *add.* denotes additions of matrices or vectors. Our formulation needs consistently less operations than the baseline approach. The blue numbers give the number of operations for a cubic spline ($k=4$).

we now show that $\widetilde{m}_{0,n}^{(k+1)} = \delta_{n,0}$ for $n = 0, ..., k-1$. If not indicated otherwise, we use well-known binomial sum properties, as summarized in e.g. [10, 0.15]. As a first step, we use the property $C_{k+1}^{l} = C_k^l + C_k^{l-1}$ and split the terms in the double sum to obtain

$$\sum_{s=0}^{k}\sum_{l=0}^{s}(-1)^l C_{k+1}^l (s-l)^{k-n} = T_1+T_2+T_3+T_4\,, \tag{80}$$

with

$$T_1 = \sum_{s=0}^{k-1}\sum_{l=0}^{s}(-1)^l C_k^l (s-l)^{k-n}\,, \tag{81}$$

$$T_2 = \sum_{l=0}^{k}(-1)^l C_k^l (k-l)^{k-n}\,, \tag{82}$$

$$T_3 = \sum_{s=0}^{k}\sum_{l=0}^{s-1}(-1)^l C_k^{l-1} (s-l)^{k-n}\,, \tag{83}$$

$$T_4 = \sum_{s=0}^{k}(-1)^s C_k^{s-1} (s-s)^{k-n}\,. \tag{84}$$

It is easy to see from (79) that $T_1 = (k-1)!\,\delta_{n,1}$. Furthermore, $T_2 = k!\,\delta_{n,0}$. For T_4, we have

$$\begin{aligned} T_4 &= 0^{k-n}\sum_{s=0}^{k}(-1)^s C_k^{s-1} = \delta_{n,k}\sum_{s=0}^{k-1}(-1)^{s+1}C_k^s \\ &= -\delta_{n,k}(-1)^{k-1}C_{k-1}^{k-1} = \delta_{n,k}(-1)^k\,. \end{aligned} \tag{85}$$

Finally, we need T_3, which is the most complicated term:

$$\begin{aligned} T_3 &= \sum_{s=0}^{k}\sum_{l=0}^{s-1}(-1)^l C_k^{l-1}(s-l)^{k-n} \\ &= \sum_{s=0}^{k-1}\sum_{l=0}^{s-1}(-1)^{l+1} C_k^{l}(s-l)^{k-n} \\ &= -\sum_{s=0}^{k-1}\sum_{l=0}^{s}(-1)^{l} C_k^{l}(s-l)^{k-n} + \sum_{s=0}^{k-1} C_k^s 0^{k-n} \\ &= -(k-1)!\,\delta_{n,1} + \delta_{n,k}(-1)^{k-1}\,, \end{aligned} \tag{86}$$

where the first equality comes from index shifting ($s \leftarrow s-1$ and $l \leftarrow l-1$), and the last one uses the induction assumption. In total, we obtain:

$$\begin{aligned} & T_1+T_2+T_3+T_4 \\ &= (k-1)!\,\delta_{n,1} + k!\,\delta_{n,0} \\ &\quad - (k-1)!\,\delta_{n,1} - \delta_{n,k}(-1)^k + \delta_{n,k}(-1)^k \\ &= k!\,\delta_{n,0}\,, \end{aligned} \tag{87}$$

which is equivalent to $\widetilde{m}_{0,n}^{(k+1)} = \delta_{n,0}$ for $n = 0, ..., k-1$ by definition of $\widetilde{m}_{j,n}^{(k+1)}$.

What remains is the case $n = k$:

$$\sum_{s=0}^{k}\sum_{l=0}^{s}(-1)^l C_{k+1}^l (s-l)^0 = \sum_{s=0}^{k}(-1)^s C_k^s = 0\,, \tag{88}$$

which concludes the proof.

8.5.2 Proof of (51) (Jacobian of $\mathrm{Exp}(-\lambda\mathbf{d})$ multiplied by a vector)

$$\begin{aligned} &\frac{\partial}{\partial \mathbf{d}}\,\mathrm{Exp}(-\lambda\mathbf{d})\boldsymbol{\omega} = \frac{\partial}{\partial \boldsymbol{\delta}}\,\mathrm{Exp}(-\lambda(\mathbf{d}+\boldsymbol{\delta}))\boldsymbol{\omega}\Big|_{\boldsymbol{\delta}=0} \\ &= \frac{\partial}{\partial \boldsymbol{\delta}}\left(\mathrm{Exp}(-\lambda\mathbf{d})\,\mathrm{Exp}(-J_\mathrm{r}(-\lambda\mathbf{d})\lambda\boldsymbol{\delta})\boldsymbol{\omega} + \mathcal{O}(\boldsymbol{\delta}^2)\right)\Big|_{\boldsymbol{\delta}=0} \\ &= -\lambda\,\mathrm{Exp}(-\lambda\mathbf{d})\,\frac{\partial}{\partial \boldsymbol{\delta}}(\mathrm{Exp}(\boldsymbol{\delta})\boldsymbol{\omega})\Big|_{\boldsymbol{\delta}=0} J_\mathrm{r}(-\lambda\mathbf{d}) \\ &= \lambda\,\mathrm{Exp}(-\lambda\mathbf{d})\boldsymbol{\omega}_\wedge J_\mathrm{r}(-\lambda\mathbf{d})\,. \end{aligned} \tag{89}$$

For the second equality we have used the right Jacobian property (10). To obtain the last equality, note that

$$\frac{\partial\,\mathrm{Exp}(\boldsymbol{\delta})\boldsymbol{\omega}}{\partial \boldsymbol{\delta}}\Big|_{\boldsymbol{\delta}=0} = -\boldsymbol{\omega}_\wedge\,. \tag{90}$$

8.5.3 Proof of (59) (Jacobian of acceleration)

We show by induction that the following two formulas are equivalent for $l = j+2, \ldots, k$:

$$\frac{\partial \dot{\boldsymbol{\omega}}^{(l)}}{\partial \mathbf{d}_j} = -\dot{\lambda}_{l-1}D_{l-1}\frac{\partial \boldsymbol{\omega}^{(l)}}{\partial \mathbf{d}_j} + A_{l-1}^\top \frac{\partial \dot{\boldsymbol{\omega}}^{(l-1)}}{\partial \mathbf{d}_j}\,, \tag{91}$$

$$\frac{\partial \dot{\boldsymbol{\omega}}^{(l)}}{\partial \mathbf{d}_j} = P_j^{(l)}\frac{\partial \dot{\boldsymbol{\omega}}^{(j+1)}}{\partial \mathbf{d}_j} - (\mathbf{s}_j^{(l)})_\wedge \frac{\partial \boldsymbol{\omega}^{(l)}}{\partial \mathbf{d}_j}\,, \tag{92}$$

where we define $P_j^{(l)}$ and $\mathbf{s}_j^{(l)}$ as

$$P_j^{(l)} = \left(\prod_{m=j+1}^{l-1} A_m \right)^{\top} \quad \Rightarrow \quad P_j^{(k)} = P_j \, , \tag{93}$$

$$\mathbf{s}_j^{(l)} = \sum_{m=j+1}^{l-1} \dot{\lambda}_m P_m \mathbf{d}_m \quad \Rightarrow \quad \mathbf{s}_j^{(k)} = \mathbf{s}_j \, . \tag{94}$$

The case $l = k$ then is the desired results. For $l = j + 2$, we easily see that

$$-\dot{\lambda}_{l-1} D_{l-1} = -(\dot{\lambda}_{j+1} \mathbf{d}_{j+1})_{\wedge} = -(\mathbf{s}_j^{(l)})_{\wedge} \, , \tag{95}$$

$$A_{l-1}^{\top} = A_{j+1}^{\top} = P_j^{(j+2)} = P_j^{(l)} \, , \tag{96}$$

which together implies the equivalence of (91) and (92). Now, assume the equivalence holds for some $l \in \{j + 2, \ldots, k-1\}$ and note that

$$A_l^{\top} P_j^{(l)} = P_j^{(l+1)} \, , \tag{97}$$

$$A_l^{\top} (s_j^{(l)})_{\wedge} = \left((s_j^{(l+1)})_{\wedge} - \dot{\lambda}_l D_l \right) A_l^{\top} \, . \tag{98}$$

Then, starting from (91) and using the induction assumption, we obtain

$$\begin{aligned} \frac{\partial \dot{\boldsymbol{\omega}}^{(l+1)}}{\partial \mathbf{d}_j} &= -\dot{\lambda}_l D_l \frac{\partial \boldsymbol{\omega}^{(l+1)}}{\partial \mathbf{d}_j} \\ &\quad + A_l^{\top} \left(P_j^{(l)} \frac{\partial \dot{\boldsymbol{\omega}}^{(j+1)}}{\partial \mathbf{d}_j} - (\mathbf{s}_j^{(l)})_{\wedge} \frac{\partial \boldsymbol{\omega}^{(l)}}{\partial \mathbf{d}_j} \right) \\ &= -\dot{\lambda}_l D_l \frac{\partial \boldsymbol{\omega}^{(l+1)}}{\partial \mathbf{d}_j} + P_j^{(l+1)} \frac{\partial \dot{\boldsymbol{\omega}}^{(j+1)}}{\partial \mathbf{d}_j} \\ &\quad - \left((s_j^{(l+1)})_{\wedge} - \dot{\lambda}_l D_l \right) A_l^{\top} \frac{\partial \boldsymbol{\omega}^{(l)}}{\partial \mathbf{d}_j} \, . \end{aligned} \tag{99}$$

The first and the last summand cancel, and what remains is (92) for $l + 1$, which concludes the proof.